Because of the agreement between the laminar and measured turbulent boundary-layer profiles there is merit in indicating parametrically the effects of flow acceleration and thermal boundary condition on laminar boundary layers since the same trends may also be found in turbulent boundary layers in the region beyond the viscous sublayer. These predicted profiles are shown in Fig. 3. Even for a small value of $\bar{\beta}$ of 0.5, there is a noticeable difference from the Crocco relation. The profile at $\bar{\beta}$ of 20 would nearly correspond to the throat condition for the nozzle investigated (Fig. 1), but with a larger amount of wall cooling, i.e., $T_w/T_{t0} = 0.2$. With more wall cooling the profiles depart less from the Crocco relationship; the opposite is true for wall heating for which case the velocity profiles have overshoot, e.g., see Cohen and Reshotko.[11]

### Summary and Conclusions

The measured relationship between the total temperature and velocity profiles in an accelerating turbulent boundary-layer flow through a nozzle was found to differ from the Crocco relation, with the amount of departure being dependent upon an acceleration parameter $\bar{\beta}$ which is related to nozzle shape. For accelerating flow the temperature-velocity relationship in the region beyond the viscous sublayer was similar to that for a laminar boundary layer and because of this correspondence, the eddy diffusivities for heat and momentum transfer $\epsilon_m$ and $\epsilon_h$ at each probe location are believed not to have differed much from each other. Based on the agreement between the laminar and measured turbulent boundary-layer profiles beyond the viscous sublayer, a rather large departure from the Crocco relation is expected because of flow acceleration and the thermal boundary condition, i.e., cooling or heating. From these observations it does appear that previous investigations of the structure of turbulent boundary layers made near the exit of nozzles could have been influenced to some extent by the pressure gradient that still existed at the location of interest and/or by the upstream history of the flow through the nozzle.

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# Parametric Differentiation Method to Structural Optimization Problems


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A STRUCTURAL optimization problem can be cast as a mathematical programing problem in the form,

$$\min f(\mathbf{X}) \tag{1}$$

subject to

$$g_j(\mathbf{X}) \leq 0 \quad j = 1, \ldots, m$$

where $\mathbf{X}$ is a $n$-dimensional vector of design variables $x_i$, $i = 1, 2, \ldots, n$ and $g_j(\mathbf{X})$ are the given constraints on the design. The function $f(\mathbf{X})$ is called the objective function and its choice is governed by the nature of the problem. One of the methods to obtain the solution of the constrained minimization problem Eq. (1) is to solve a sequence of unconstrained minimization problems of the form,

$$\Phi(\mathbf{X},r) = f(\mathbf{X}) - r \sum_{j=1}^{m} \frac{1}{g_j(\mathbf{X})} \tag{2}$$

where $\Phi$ is the penalty function and $r$ is an *arbitrary* penalty parameter which in the limit goes to zero. Solution to several structural optimization problems[1,2] has been successfully obtained by the sequential minimization of the penalty function $\Phi$. However, the discrete numerical values chosen for the penalty parameter $r$ influences the rate of convergence of this method.[3,4]

This Note presents a parametric differentiation approach to obtain the solution of Eq. (1). This is achieved through the minimization of the function $\Phi(\mathbf{X},r)$ where $\mathbf{X}$ is now treated to be a function of the continuous parameter $r$. The unconstrained minimization problem is transformed by the classical calculus approach to a set of simultaneous nonlinear algebraic equations. The latter, in turn is transformed to a set of simultaneous linear differential equations with variable coefficients. The resulting initial value problem in ordinary differential equations can be solved numerically to find $\mathbf{X}(r)$ for $r = 0$ in order to give the solution of Eq. 1. The convergence to the solution is geared to the uniqueness of the associated first order initial value problem and thus is guaranteed. This method ensures that the solution so obtained is only a stationary point. In order to satisfy that the solution is a minimum of Eq. (2), the matrix of second partials of $\Phi$ function at the solution should be positive definite. No additional efforts are required to formulate the matrix of the second partials of $\Phi$. This is obtained from a part of the algorithm proposed here. A method is indicated to test the positive definiteness of this matrix which ensures the solution to Eq. (1).

The numerical experimentation on the proposed method for structural optimization problems is in progress.

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### Mathematical Description

For any positive value of $r = \bar{r}$, the necessary condition for the function $\Phi(\mathbf{X},\bar{r})$ to be minimum is

$$\partial\Phi(\mathbf{X},\bar{r})/\partial x_i = 0 \qquad i = 1,2, \ldots, n \quad (3)$$

From Eqs. (2) and (3) we get

$$\frac{\partial f(\mathbf{X})}{\partial x_i} + \bar{r} \sum_{j=1}^{m} \frac{1}{g_j{}^2} \frac{\partial g_j(\mathbf{X})}{\partial x_i} = 0 \qquad i = 1,2, \ldots, n \quad (4)$$

Equation (4) forms a set of $n$ simultaneous nonlinear algebraic equations in $n$ unknowns $\mathbf{X}(r)$. Starting from a feasible design $\mathbf{X}_0$ and considering a parameter $\lambda$ such that $0 \leq \lambda \leq 1$ we can embed Eq. (4) in the form,

$$\frac{\partial f(\mathbf{X})}{\partial x_i} + \bar{r} \sum_{j=1}^{m} \frac{1}{g_j{}^2} \frac{\partial g_j(\mathbf{X})}{\partial x_i} = (1-\lambda)\left[\frac{\partial f(\mathbf{X})}{\partial x_i} + \bar{r} \sum_{j=1}^{m} \frac{1}{g_j{}^2} \frac{\partial g_j(\mathbf{X})}{\partial x_i}\right]_{\mathbf{X}=\mathbf{X}_0} \quad (5)$$
$$i = 1,2, \ldots, n$$

Equation (5) has a characteristic that at $\lambda = 0$, $\mathbf{X}(\bar{r}) = \mathbf{X}_0$ and at $\lambda = 1$, $\mathbf{X}(\bar{r})$ is a solution to Eq. (4). Any other solution of Eq. (5) is $\mathbf{X}(\bar{r},\lambda)$.

Differentiating Eq. (5) with respect to $\lambda$ and using the chain rule, we obtain

$$\sum_{k=1}^{n} \frac{\partial^2 f(\mathbf{X})}{\partial x_k \partial x_i} \cdot \frac{dx_k}{d\lambda} + \bar{r} \sum_{j=1}^{m} \left[\frac{1}{g_j{}^2} \sum_{k=1}^{n} \frac{\partial^2 g_j(\mathbf{X})}{\partial x_k \partial x_i} \cdot \frac{dx_k}{d\lambda} - \frac{2}{g_j{}^3} \cdot \frac{\partial g_j(\mathbf{X})}{\partial x_i} \sum_{k=1}^{n} \frac{\partial g_j(\mathbf{X})}{\partial x_k} \cdot \frac{dx_k}{d\lambda}\right] =$$
$$-\left[\frac{\partial f(\mathbf{X})}{\partial x_i} + \bar{r} \sum_{j=1}^{m} \frac{1}{g_j{}^2} \cdot \frac{\partial g_j(\mathbf{X})}{\partial x_i}\right]_{\mathbf{X}=\mathbf{X}_0} \qquad i = 1,2, \ldots, n \quad (6)$$

Writing Eq. (6) in matrix form, we have

$$[H]\left\{\frac{d\mathbf{X}}{d\lambda}\right\} = -\begin{Bmatrix} \frac{\partial\Phi}{\partial x_1}(\mathbf{X}_0,\bar{r}) \\ \frac{\partial\Phi}{\partial x_2}(\mathbf{X}_0,\bar{r}) \\ \vdots \\ \frac{\partial\Phi}{\partial x_n}(\mathbf{X}_0,\bar{r}) \end{Bmatrix} \quad (7)$$

where $[H]$ is a $n \times n$ matrix whose element $h_{ik}$ is given by

$$h_{ik} = \frac{\partial^2 f(\mathbf{X})}{\partial x_k \partial x_i} + \bar{r} \sum_{j=1}^{m} \left[\frac{1}{g_j{}^2} \cdot \frac{\partial^2 g_j(\mathbf{X})}{\partial x_k \partial x_i} - \frac{2}{g_j{}^3} \frac{\partial g_j}{\partial x_i} \cdot \frac{\partial g_j}{\partial x_k}\right] \quad (8)$$

Clearly, the matrix $[H]$ is a matrix of second partial of the function $\Phi(\mathbf{X},\bar{r})$.

Equation (7) represents a set of $n$ simultaneous ordinary differential equations which can be solved numerically for $dx_l/d\lambda$; $l = 1, 2, \ldots, n$. The resulting ordinary differential equations with initial conditions $\mathbf{X} = \mathbf{X}_0$ at $\lambda = 0$ can be numerically integrated by any standard scheme, e.g., Gill variation of the Runge-Kutta formula,[5] up to $\lambda = 1$ to give the solution $\mathbf{X}(\bar{r})$. The step length (not necessarily constant) for integration will be governed by the accuracy required and the speed of calculation. It is to be noticed that the elements of $[H]$ matrix will have to be computed at every step of the numerical integration process.

Having obtained a solution $\mathbf{X}(\bar{r})$ of Eq. (2), the result $\mathbf{X}(r)$ for $r = 0$, which is the solution of Eq. (1), can be obtained by almost a repetition of the aforementioned procedure as now described.

Differentiating Eq. (2) with respect to design variables $x_i$, $i = 1,2, \ldots, n$, and in order to satisfy the necessary condition for minima of $\Phi$, setting them equal to zero we get,

$$\frac{\partial f(\mathbf{X})}{\partial x_i} + r \sum_{j=1}^{m} \frac{1}{g_j{}^2}\left[\frac{\partial g_j(\mathbf{X})}{\partial x_i}\right] = 0; \quad i = 1,2, \ldots, n \quad (9)$$

Again, differentiating Eq. (9) with respect to parameter $r$, using the chain rule and representing the result in the matrix form, we obtain

$$[H]\left\{\frac{d\mathbf{X}}{dr}\right\} = -\begin{Bmatrix} \sum_{j=1}^{m} \frac{1}{g_j{}^2} \frac{\partial g_j(\mathbf{X})}{\partial x_1} \\ \sum_{j=1}^{m} \frac{1}{g_j{}^2} \frac{\partial g_j(\mathbf{X})}{\partial x_2} \\ \vdots \\ \sum_{j=1}^{m} \frac{1}{g_j{}^2} \frac{\partial g_j(\mathbf{X})}{\partial x_n} \end{Bmatrix} \quad (10)$$

where the elements of the matrix $[H]$ are given by Eq. (8) with $\bar{r}$ substituted by $r$. The solution to Eq. (10) can be obtained in a manner similar to the one described for the solution of Eq. (7). Now the initial conditions are $\mathbf{X} = \mathbf{X}(r)$ at $r = \bar{r}$ which are known.

The solution $\mathbf{X}(0)$ of (2) corresponds to a minimum if and only if the matrix $[H]$ is positive definite. The positive definiteness of the matrix $[H]$ at $\mathbf{X} = \mathbf{X}(0)$ can be easily tested. Since $[H]$ is a symmetric matrix, it can be uniquely decomposed[6] to a lower and an upper triangular matrix which are transpose of each other.

$$[H] = [L][L]^T \quad (11)$$

$$\{\mathbf{X}\}^T[H]\{\mathbf{X}\} = \{\mathbf{X}\}^T[L][L]^T\{\mathbf{X}\} = [[L]^T\{\mathbf{X}\}]^T[L]^T\{\mathbf{X}\} \geqslant 0 \text{ for all } \{\mathbf{X}\} \text{ and will be zero only if } \{\mathbf{X}\} = \{0\}$$

If the elements of the matrix $[L]$ are all real numbers, then $[H]$ is a positive definite matrix as seen from the following:

$$\{\mathbf{X}\}^T[H]\{\mathbf{X}\} = \{\mathbf{X}\}^T[L][L]^T\{\mathbf{X}\} = [[L]^T\{\mathbf{X}\}]^T[L]^T\{\mathbf{X}\} \geqslant 0 \text{ for all } \mathbf{X} \text{ and will be zero only if } \{\mathbf{X}\} = \{0\}$$

### Concluding Remarks

The proposed algorithm seems to be computationally tractable for structural optimization problems. It is assumed that the first and second order derivatives of the objective function $f(\mathbf{X})$ and the constraint functions $g_j(\mathbf{X})$ are available in some form. It is possible to obtain the exact first and second order derivatives of constraint quantities such as stress, displacement, frequency[7,8] etc. with respect to design variables. However, if the functions are so complicated that the exact derivatives with respect to design variables are not possible, then finite difference approach may be used. The optimization problem is reduced to the solution of a series of simultaneous nonlinear algebraic equations. Set of simultaneous nonlinear algebraic equations have been efficiently solved by the parametric differentiation approach.[9-11] A computationally efficient method of testing the positive definiteness of the matrix of second partial has been indicated. The need for the arbitrary choice of the penalty parameter, $r$, in the Fiacco McCormick algorithm is eliminated. The choice of the starting point $X_0$ is not absolutely arbitrary. Any feasible design $X_0$ corresponding to which the matrix $H$ is positive definite will presumably confine the path at least in the local valley and as, $r$, tends to zero will lead to the local minimum. The proposed method is by no chance restricted to the solution of structural optimization problems and, in fact, can be used to obtain the solution of nonlinear mathematical programing problems.

After the completion of the present work, the authors have come to know of a similar approach briefly suggested in Ref. 12.

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# Axisymmetric Ablation with Shape Changes and Internal Heat Conduction


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### Nomenclature

$c_p$ = specific heat at constant pressure
$E$ = defined in Eq. (19)
$F$ = energy required to ablate unit mass of body
$H$ = total enthalpy
$k$ = thermal conductivity
$L$ = body length (Fig. 1)
$p$ = pressure
$q_i$ = heat flux across inside surface
$q_o$ = heat flux across outside surface
$r$ = radial position (Fig. 1)
$R$ = radial coordinate of the cut-off point (Fig. 1)
$R_n$ = nose radius $= (-r_{st}''')^{-1/2}$
$s$ = recession depth in axial direction
$t$ = time
$T$ = temperature at a position $(r, \phi, t)$
$T_i$ = initial temperature of body
$u$ = velocity
$x$ = distance along the curved surface
$z$ = axial position (Fig. 1)
$z_i$ = axial position of original surface at fixed radial position (Fig. 1)
$\alpha$ = thermal diffusivity
$\beta$ = angle between local surface tangent in meridian plane and axis of symmetry
$\gamma$ = ratio of specific heats
$\delta$ = cone half-angle
$\mu$ = viscosity
$\xi$ = defined in Eq. (10)
$\rho$ = material density of the ablating body
$\rho_{st}$ = shocked air density at stagnation point
$\phi$ = transformed geometric variable (Eq. (1))
$\omega$ = temperature exponent for viscosity variation

#### *Subscripts*

$e$ = boundary-layer edge condition
$n$ = components normal to local surface
st = stagnation point
$\infty$ = ambient conditions
$w$ = surface
$'$ = $d/dx$


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## I. Introduction

THIS paper presents the effects of internal heat conduction and shape change on the calculation of the ablation of a metal nose tip. This combined problem has not been considered previously. For example, Benjamin[1] introduces the effects of the shape change on the stagnation-point heat flux for sphere-cones in the form of correction factors to the results one would obtain without the effects of geometry change on the heat flux. Transverse heating effects in Ref. 1 are not considered; instead, an effective heat of ablation is introduced. Diffusion of heat in the direction tangential to the surface is also neglected in Ref. 2 compared with that in the direction normal to the surface.

In this paper, a theoretical approach is presented which couples the instantaneous body geometry with the external heating environment and considers the internal flow of heat by thermal heat conduction for blunt-nosed, axially symmetric bodies at hypersonic speeds. Local heat-transfer rates are calculated to correspond to instantaneous boundary-layer growth which varies with flight trajectory and changing body shape. A Landau transformation modified for axisymmetric bodies is used to allow for a receding surface in a three-dimensional coordinate system, such that one of its coordinates remains fixed at the moving surface (Fig. 1).

The general method is applicable whether the ablation is due to melting, sublimation or surface reaction. Results for a

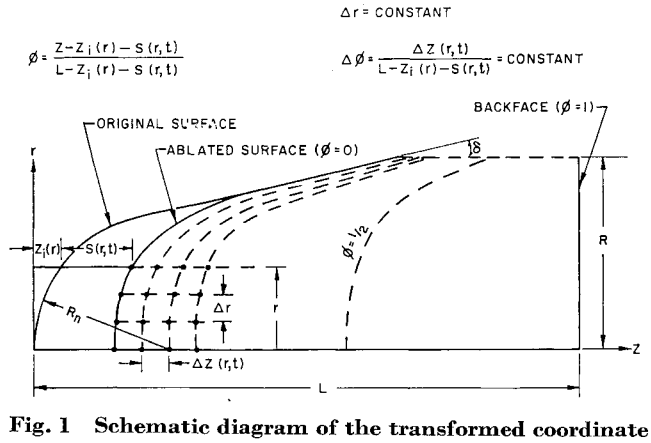


**Fig. 1 Schematic diagram of the transformed coordinate system.**